FLOW OF PLASTICO-VISCOUS MEDIA IN NONCIRCULAR PIPES

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The flow of a plastico-viscous medium under a constant pressure head in circular and annular pipes was considered in papers [1–4].

In this paper we use the method of a small parameter to study the steady flow of a plastico-viscous medium in noncircular pipes.

1. In an infinitely long elliptical pipe let

$$\frac{x^2}{b^2(1+\delta)^2}+\frac{y^2}{b^2(1-\delta)^2}=1 \qquad (\delta<1),$$

where δ is a dimensionless parameter; the material flows in the positive direction of the z axis under the action of a constant pressure gradient $q^2 = -\partial p/\partial z$.

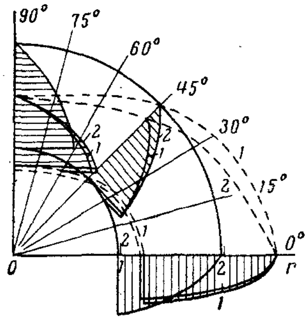

Fig. 1

We denote by μ the viscosity coefficient, by k the yield point, by S the cross-sectional area, and by L the perimeter of the core cross section.

Assuming that the flow is linear, we have in a cylindrical coordinate system [2]

$$\left(\mu+\frac{\tau_0}{h}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)-\frac{\tau_0}{h^2}\left(\frac{\partial h}{\partial r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial h}{\partial\varphi}\frac{\partial w}{\partial\varphi}\right)=-q^2$$

$$h=\left[\left(\frac{\partial w}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial w}{\partial\varphi}\right)^2\right]^{1/2}. \qquad (1.1)$$

Here $w(r, \varphi)$ is the velocity and τ_0 is the limiting shear stress.

For determining $w(r, \varphi)$, the unknown boundary $r_s = r_s(\varphi)$ of the rigid core and its velocity v = const, it is necessary to solve Eq. (1.1) for the following conditions:

$$\partial w/\partial n=0, \quad q^2 S=kL, \quad v=w \quad \text{for } r=r_s. \qquad (1.2)$$

Here n is the normal to the core boundary. At the surface of the pipe we assume that w = 0.

We seek a solution in the form of expansions in the small parameter

$$w=\sum_{n=0}^{\infty}\delta^n w^{(n)}, \qquad v=\sum_{n=0}^{\infty}\delta^n v^{(n)}, \qquad r_s=\sum_{n=0}^{\infty}\delta^n r_s^{(n)}. \qquad (1.3)$$

We adopt the solution of [2] for flow in a circular pipe of radius b as the zero approximation

$$w^\circ=\frac{1}{2b}(b^2-r^2)-\frac{1}{b}(b-r), \quad r_s^\circ=1, \quad v^\circ=\frac{(b-1)^2}{2b} \quad (1\leqslant r\leqslant b). \qquad (1.4)$$

Here and in what follows, we retain the earlier notation and refer the velocities $w^{(n)}$, $v^{(n)}$ to the quantity kb/μ, the polar radius r and all linear dimensions to the quantity $2k/q^2$, and the stress τ_0 to the quantity k.

The equation of the contour of the pipe cross section in cylindrical coordinates can be written in series form:

$$r=b+\delta b\cos 2\varphi-{}^3/_4\delta^2 b(1-\cos 4\varphi)+\ldots. \qquad (1.5)$$

After linearizing (1.1) and (1.2) with respect to the small parameter δ, we obtain, using (1.3)–(1.5), the equations of the first approximation

$$\frac{\partial^2 w'}{\partial r^2}+\frac{1}{r}\frac{\partial w'}{\partial r}+\frac{1}{r(r-1)}\frac{\partial^2 w'}{\partial\varphi^2}=0, \qquad (1.6)$$

$$r_s'=b\left.\frac{\partial w'}{\partial r}\right|_{r=1}, \quad v'=w'|_{r=1}, \quad \int_0^{2\pi} r_s' d\varphi=0,$$

$$\left.\frac{\partial w'}{\partial\varphi}\right|_{r=1}=0, \quad w'|_{r=b}=(b-1)\cos 2\varphi.$$

Representing the solution of Eq. (1.6) in the form $w' = R(r)\Phi(\varphi)$, we obtain

$$w'=C_0\ln r+D_0[C_1 f_1(r)+D_1 f_2(r)]\cos 2\varphi, \quad (C_0,\ D_0,\ C_1,\ D_1=\text{const}),$$

where

$$f_1(r)=r^2-\frac{4}{3}r+\frac{1}{3},$$

$$f_2(r)=\left[9\ln r-20.25\frac{1}{3r-1}-9\ln(r-1)-\frac{2{,}25}{r-1}\right]\times$$

$$\times(r-1)\left(r-\frac{1}{3}\right)$$

are the solutions of the hypergeometric equation

$$r(r-1)\frac{d^2R}{dr^2}+(r-1)\frac{dR}{dr}-4R=0.$$

From the boundary conditions (1.6) we obtain $D_0 = 0$, $C_0 = 0$, $D_1 = 0$, $C_1 = 3/(3b - 1)$. Thus,

$$w'=\frac{3r^2-4r+1}{3b-1}\cos 2\varphi, \quad r_s'=\frac{2b}{3b-1}\cos 2\varphi, \qquad v'=0. \qquad (1.7)$$

The equations for the second approximation of the problem are

$$\frac{\partial^2 w''}{\partial r^2}+\frac{1}{r}\frac{\partial w''}{\partial r}+\frac{1}{r(r-1)}\frac{\partial^2 w''}{\partial\varphi^2}=\Psi(r)+\Theta(r)\cos 4\varphi,$$

$$\int_0^{2\pi} r_s'' d\varphi=\frac{4b^2\pi}{(3b-1)^2}, \quad w''|_{r=b}=\frac{19b-18b^2-3}{4(3b-1)}+\frac{3-5b}{4(3b-1)}\cos 4\varphi,$$

$$\left.\frac{\partial w''}{\partial\varphi}\right|_{r=1}=\frac{4b}{(3b-1)^2}\sin 4\varphi, \qquad (1.8)$$

$$r_s''=b\left.\frac{\partial w''}{\partial r}\right|_{r=1}+\frac{12b}{(3b-1)^2}\cos^2 2\varphi, \quad v''=w''|_{r=1}+\frac{b}{(3b-1)^2}(1+\cos 4\varphi),$$

$$\Psi(r)=\frac{b(1-9r^2)}{r^3(3b-1)^2}, \qquad \Theta(r)=\frac{b(3r-1)(27r^2-18r-1)}{r^3(r-1)(3b-1)^2}.$$

Separating the variables, we obtain

$$w''=R_1(r)+R_2(r)\cos 4\varphi$$

where $R_1(r)$ and $R_2(r)$ are the general solutions of the inhomogeneous equations

$$\frac{d^2R_1}{dr^2}+\frac{1}{r}\frac{dR_1}{dr}=\Psi(r), \quad \frac{d^2R_2}{dr^2}+\frac{1}{r}\frac{dR_2}{dr}-\frac{16}{r(r-1)}R_2=\Theta(r). \qquad (1.9)$$

The general solution of the first equation of (1.9) is

$$R_1 = -\frac{b}{(3b-1)^2}\left(A\ln r + 9r - \frac{1}{r} + B\right) \qquad (A,\ B = \mathrm{const}).$$

The second equation of (1.9) is a hypergeometric equation and its solution [5] is

$$R_2 = C_2P(r) + C_3T(r) + \frac{b}{(3b-1)^2}E(r), \quad (C_2,\ C_3 = \mathrm{const}),$$

$$P(r) = 35r^4 - 80r^3 + 60r^2 - 16r + 1,$$

$$T(r) = P(r)\left[\ln r + \sum_{i=1}^{4} F_i \ln(r-a_i) - \sum_{i=1}^{4} H_i (r-a_i)^{-1}\right],$$

$$E(r) = P(r)\left[32\ln r - \frac{1}{r} + \sum_{i=1}^{4} S_i \ln(r-a_i) - \sum_{i=1}^{4} M_i (r-a_i)^{-1}\right],$$

$$\begin{array}{llll} a_1 = 1, & a_2 = 0.78775, & a_3 = 0.08870, & a_4 = 0.40926, \\ F_1 = -1, & F_2 = 0.002646, & F_3 = 0.0049553, & F_4 = -0.0052577, \\ H_1 = 0.0625, & H_2 = 0.32857, & H_3 = 0.22074, & H_4 = 0.38794, \\ S_1 = -32, & S_2 = 0.10536, & S_3 = 0.1828, & S_4 = -0.16707, \\ M_1 = 2.25, & M_2 = 10.44778, & M_3 = 6.17281, & M_4 = 12.11781. \end{array}$$

Using the boundary conditions (1.8), we finally obtain

$$w'' = -\frac{b}{(3b-1)^2}\left[A\ln r + 9r - \frac{1}{r} + B\right] + \left[C_2P(r) + C_3T(r) + \frac{bE(r)}{(3b+1)^2}\right]\cos 4\varphi, \tag{1.10}$$

$$A = -6,\quad B = \frac{-342b^3 + 271b^2 - 63b + 4 - 24b^2(3b-1)\ln b}{4b(3b-1)^2},$$

$$C_3 = -\frac{32b}{(3b-1)^2},\quad C_2 = \frac{(14b - 15b^2 - 3) + 128bT(b) - 4bE(b)}{4(3b-1)^2P(b)}.$$

From (1.10) and (1.18) it is easy to obtain r_S'' and v'', whose expressions we omit because of their excessive length.

It should be noted that from the condition of existence of flow in an elliptical pipe $q^2S_* \geq KL_*$ (S_* is the area and L_* the perimeter of the elliptical cross section of the pipe) it follows that $b \geq 1 + 1.25^2$. For $b < 1 + 1.25\,\delta^2$ the pressure head q^2 is insufficient to produce a flow.

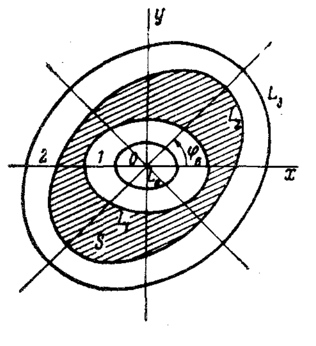

Fig. 2

Figure 1 gives the contour of the pipe, the boundaries of the rigid core and the velocity curves in the first and second approximations for the case b = 2 and δ = 0.25. The broken lines show the contours of the core and the cross section of the pipe in the first and second approximations. The velocity curve of the zero approximation agrees with the velocity curve of the first approximation for $\varphi = 45°$.

In Fig. 1 the dimensions of the velocity curves are increased five times.

2. Let us now consider the steady flow of a plastico-viscous medium in the space between two coaxial elliptical pipes. We assume that the axes of the elliptical contours L_0 and L_3 of the internal and the external pipes form an angle φ_0 (Fig. 2), and that under the action of a constant pressure gradient $q^2 = -\partial p/\partial z$ the medium has a velocity $w(r,\ \varphi)$ in the positive direction of the z axis. We denote by L_0, L_3 the perimeters of the internal and external pipes, by L_1, L_2 the perimeters of the internal and external boundaries of the rigid core and by S the cross-sectional area of the core.

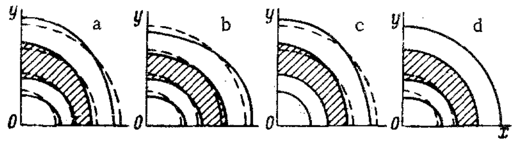

Fig. 3

For determining $w(r,\ \varphi)$, the equations for $r_1(\varphi)$, $r_2(\varphi)$, i.e., the boundaries L_1 and L_2 of the core, and its velocity v, we have Eq. (1.1) and the relations

$$q^2S = k(L_1 + L_2),\quad w(r_1,\ \varphi) = w(r_2,\ \varphi) = v,$$

$$\left.\frac{\partial w}{\partial n}\right|_{L_1} = \left.\frac{\partial w}{\partial n}\right|_{L_2} = 0,\quad w|_{L_1} = w|_{L_2} = 0. \tag{2.1}$$

We will use the dimensionless quantity δ < 1, which characterizes the ellipticity of the contours L_0 and L_3, as the small parameter. Representing the unknown quantities w, r_1, r_2, and v as series (1.3), we retain only the terms of order δ

$$w = w^\circ + \delta w',\quad r_1 = r_1^\circ + \delta r_1',\quad r_2 = r_2^\circ + \delta r_2',\quad v = v^\circ + \delta v'. \tag{2.2}$$

The equations of the boundaries L_0 and L_3 can be written respectively in the following form:

$$r_0 = a(1 + \delta d_0\cos 2\varphi),\quad r_3 = b\,[1 + \delta d_3\cos 2(\varphi - \varphi_0)], \tag{2.3}$$

where a and b are the radii of the circles into which the ellipses L_0 and L_3 transform at δ = 0, and d_0 and d_3 are dimensionless coefficients.

Retaining the earlier notation, we change to dimensionless variables by referring w°, w', v°, and v' to $2k^2/q^2\mu$, the stress τ_0 to k, and the polar radius r, a, and b to $2k/q^2$.

The solution of the problem in the zero-th approximation [3] can be written

$$w^\circ = \frac{a^2 - r^2}{2} + a - r + r_1^\circ(r_1^\circ + 1)\ln\frac{r}{a}\quad (a \leqslant r \leqslant r_1^\circ),$$
$$w^\circ = \frac{b^2 - r^2}{2} - b + r + r_2^\circ(r_2^\circ - 1)\ln\frac{r}{b}\quad (r_2^\circ \leqslant r \leqslant b), \tag{2.4}$$

where the radii r_1° and r_2° of the internal and external boundaries of the rigid core are related by the following equation:

$$r_2^\circ = r_1^\circ + 1,\quad \frac{a^2 - r_1^{\circ 2}}{2} + a - r_1^\circ + r_1^\circ(r_1^\circ + 1)\ln\frac{r_1^\circ}{a} = \frac{b^2 - r_2^{\circ 2}}{2} - b + r_2^\circ + r_2^\circ(r_2^\circ - 1)\ln\frac{r_2^\circ}{b}.$$

It should be noted that $v^\circ = w^\circ(r_1^\circ) = w^\circ(r_2^\circ)$.

Linearizing Eqs. (1.1) and (2.1) and using (2.2)-(2.4), we obtain equations for the components of the first approximation.

For the region 1 adjoining the boundary of the internal pipe (Fig. 2) we have

$$w'(a,\ \varphi) = -\left.\frac{dw^\circ}{dr}\right|_{r=a} a d_0\cos 2\varphi, \tag{2.5}$$

$$r_1' = -\left[\frac{\partial w'}{\partial r}\left(\frac{\partial^2 w^\circ}{\partial r^2}\right)^{-1}\right]_{r=r_1^\circ},\quad \left.\frac{\partial w'}{\partial \varphi}\right|_{r=r_1^\circ} = 0.$$

For region 2 the analogous expressions are

$$w'(b,\ \varphi) = -\left.\frac{dw^\circ}{dr}\right|_{r=b} b d_3\cos 2(\varphi - \varphi_0),$$

$$r_2' = -\left[\frac{\partial w'}{\partial r}\left(\frac{\partial^2 w^\circ}{\partial r^2}\right)^{-1}\right]_{r=r_2^\circ},\quad \left.\frac{\partial w'}{\partial \varphi}\right|_{r=r_2^\circ} = 0. \tag{2.6}$$

For w' we obtain

$$\frac{\partial^2 w'}{\partial r^2} + \frac{1}{r}\frac{\partial w'}{\partial r} + \frac{1}{r^2}\left(\frac{A - r^2}{A - r^2 \pm r}\right)\frac{\partial^2 w'}{\partial \varphi^2} = 0 \quad (A = r_1^\circ(r_1^\circ + 1)). \quad (2.7)$$

Here the negative sign corresponds to region 1 and the positive sign to region 2.

The linearized kinematic and dynamic conditions of motion of the core as a rigid whole become

$$w'(r_1^\circ, \varphi) = w'(r_2^\circ, \varphi) = v, \qquad \int_0^{2\pi} (r_2' - r_1')\, d\varphi = 0. \quad (2.8)$$

The solutions of Eqs. (2, 6) which we shall henceforth denote by by w_1' and w_2', are found as

$$w_1' = C_0 + E_0 \ln r + \sum_{\lambda=1}^{\infty} R_\lambda(r)\,[C_\lambda \cos\lambda\varphi + E_\lambda \sin\lambda\varphi], \quad (2.9)$$

$$w_2' = D_0 + F_0 \ln r + \sum_{m=1}^{\infty} T_m(r)\,[D_m \cos m(\varphi - \varphi_0) + F_m \sin m(\varphi - \varphi_0)],$$

where C_0, D_0, E_0, C_λ, E_λ, D_m, E_m are arbitrary constants, and λ, $m = 1, 2, 3, \ldots$.

From (2.8) we obtain the equations for R_λ and T_m

$$\frac{d^2 R_\lambda}{dr^2} + \frac{1}{r}\frac{dR_\lambda}{dr} - \frac{\lambda_2}{r^2}\left(\frac{r^2 - A}{r^2 - A + r}\right) R_\lambda = 0, \quad (2.10)$$

$$\frac{d^2 T_m}{dr^2} + \frac{1}{r}\frac{dT_m}{dr} - \frac{m^2 (r^2 - A)\, T_m}{r^2 (r^2 - A - r)} = 0$$

$$(\lambda,\ m = 1, 2, 3, \ldots).$$

Equations (2.10) are the equations of Fuchsian type with four singular points [5]. For the equations in R_λ, the singular points are zero, an infinitely distant point, $r_1^\circ = -1/2 + (1/4 + A)^{1/2}$ and $r_1^{\circ *} = -1/2 - (1/4 + A)^{1/2}$. The equations for T_m, in addition to zero and an infinitely distant point, contain singular points at $r_2^\circ = 1/2 + (1/4 + A)^{1/2}$ and $r_2^{\circ *} = 1/2 - (1/4 + A)^{1/2}$.

It should be noted that the singular points r_1° and r_2° of Eqs. (2.10) are located on the internal and external boundaries of the core in the zero approximation of the problem.

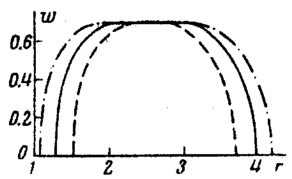

Fig. 4

Since part of the boundary conditions (2.5) and (2.6) for w' is given at $r = r_1^\circ$ and $r = r_2$, the solutions of Eqs. (2.10) in the vicinity of their singular points r_1° and r_2° should be used in (2.9).

For equations (2.10), points r_1° and r_2° are weak singular points, and the roots of the determining equations at each of them are zero and one. Therefore for each λ and m one of the independent particular solutions of each equation of (2.10) vanishes at a singular point, while the other solution, containing a logarithm, has a nonzero value at the singular point [5]. Using this and the last of relations (2.5) and (2.6), we find that in (2.10) only those particular solutions of Eqs. (2.10) that vanish at singular points r_1° and r_2° can be used.

From the boundary conditions (2.5), (2.6), and (2.8) we find that in (2.9) all the arbitrary constants with the exception of C_2 and D_2 must be made zero.

Thus, the solutions of Eqs. (2.7) for boundary conditions (2.5), (2.6), and (2.8) can be written as follows:

$$w_1' = CR(r)\cos 2\varphi, \qquad w_2' = DT(r)\cos 2(\varphi - \varphi_0). \quad (2.11)$$

We now introduce the following notation for the expansions:

$$S_\nu(r, \alpha) = \sum_{k=1}^{\infty} \alpha_{\nu k} (r - r_\nu^\circ)^{k+1}. \quad (2.12)$$

The particular solutions R(r), T(r) of Eqs. (2.10) for $\lambda = m = 2$, which are regular in the vicinity of singular points r_1° and r_2°, can be written as follows:

$$R(r) = (r - r_1^\circ) + S_1(r, C), \quad T(r) = (r - r_2^\circ) + S_2(r, C), \quad (2.13)$$

their coefficients C_{1k} and C_{2k} being successively determined from the relations [5]

$$C_{\nu n}\,[(1+n)n + a_0(1+n) + b_0] + \quad (2.14)$$

$$+ \sum_{k=1}^{n} C_{\nu,\, n-k}\,[a_{\nu k}(1 + n - k) + b_{\nu k}] = 0$$

$$(\nu = 1, 2\ n = 1, 2, 3, \ldots),$$

in which $a_{\nu k}$, $b_{\nu k}$ are coefficients of the series

$$\frac{r - r_\nu^\circ}{r} = a_{\nu 0} + (r - r_\nu^\circ)^{-1} S_\nu(r, a), \quad \frac{4(r^2 - r_1^{\circ 2} - r_1^\circ)(r_1^\circ - r)}{r^2 (r - r_\nu^\circ)} =$$

$$= b_{\nu 0} + (r - r_\nu^\circ)^{-1} S_\nu(r, b). \quad (2.15)$$

$$(\nu = 1, 2).$$

The integration constants C and D in (2.11) are given by

$$C = d_0 \frac{a^2 + a - r_1^{\circ 2} - r_1^\circ}{a - r_1^\circ + S_1(a, C)}, \qquad D = d_3 \frac{b^2 - b - r_2^{\circ 2} + r_2^\circ}{b - r_2^\circ + S_2(b, C)}. \quad (2.16)$$

In (2.13) the power series for R(r) has [5] the radius of convergence r_1°, while the power series for T(r) has the radius of convergence r_2°.

Using (2.5), (2.6), (2.4), (2.11)-(2.16), we obtain the solution of the problem in its final form:

for region 1

$$w_1' = \frac{d_0 (a^2 + a - r_1^{\circ 2} - r_1^\circ)[r - r_1^\circ + S_1(r, C)]\cos 2\varphi}{a - r_1^\circ + S_1(a, C)},$$

$$r_1' = \frac{r_1^\circ d_0 (a^2 + a - r_1^{\circ 2} - r_1^\circ)\cos 2\varphi}{(2r_1^\circ + 1)[a - r_1^\circ + S_1(a, C)]}; \quad (2.17)$$

for region 2

$$w_2' = \frac{d_3 (b^2 - b - r_2^{\circ 2} + r_2^\circ)[r - r_2^\circ + S_2(r, C)]\cos 2(\varphi - \varphi_0)}{b - r_2^\circ + S_2(b, C)},$$

$$r_2' = \frac{r_2^\circ d_3 (b^2 - b - r_2^{\circ 2} + r_1^\circ)\cos 2(\varphi - \varphi_0)}{(2r_1^\circ - 1)[b - r_2^\circ + S_2(b, C)]}. \quad (2.18)$$

Considering the linearized kinematic condition (2.8), we see that $v' = 0$.

Let us consider, as a working example, the flow of a plastico-viscous medium with $a = 1.32$, $v = 3.95$, $r_1^\circ = 2$, $r_2^\circ = 3$, $A = 6$, and $v^\circ = 0.7$.

In this case it is sufficient to retain in (2.15) only five terms of the series.

Figure 3 gives the contours of the pipe cross sections and the boundaries of the corresponding cores for $\delta = 0.25$ in four characteristic cases. Figure 3a corresponds to the case $\varphi_0 = 90°$, $d_0 = 0.7$, $d_3 = 0.25$. The case $\varphi_0 = 0$, $d_0 = 0.7$, $d_3 = 0.25$ is given in Fig. 3b. Figure 3c corresponds to values $\varphi_0 = 90°$, $d_0 = 0$, $d_3 = 0.25$. Figure 3d illustrates the case $\varphi_0 = 0$, $d_0 = 0.7$, $d_3 = 0$. The boundaries of the pipe cross sections and the core boundaries corresponding to the first approximation are given in Fig. 3 as solid lines, while the contours of the pipe cross sections and the core boundaries for the zero approximation are given as broken lines. The core cross sections corresponding to the first approximation are shaded.

Figure 4 gives the velocities in sections $\varphi = 0$ (broken line), $\varphi = 45°$ (solid line), $\varphi = 90°$ (dot-dashed line) for the case $d_0 = 0.7$, $d_3 = 0.25$, $\varphi_0 = 90°$. In the section $\varphi = 45°$, the curve for velocity w coincides with the curve for velocity $w°$ in the zero approximation.

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